Students' knowledge of Application of Mathematics – From Diagnostics to Innovations

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Abstract

The results of a questionnaire that should reveal students' knowledge about the use of computers in mathematics and the relevance of applications of mathematics in our society clearly show that current math teaching does not provide adequate ideas about the importance of computers. We describe the results and give examples of mathematical activities that are suitable to both foster mathematical concepts and widen the mathematical view. Possible changes in the curriculum are discussed.

Introduction

Computers are no longer a new ingredient in math education. The majority of German students have at least some experience with learning mathematics with computers in the classroom. However, the impact of computers on our society is much broader than just as a tool for learning. It is a tool for calculating things that would not (or not so easily) be computable without and this turns more and more disciplines into computing disciplines. Do students in school learn anything about this?

The questionnaire study, its Interpretation and fundamental ideas

In joint work with Markus Vogel (University of Education, Heidelberg) we investigated computer related knowledge of teacher students. The students were in the second half of their second year.

Most of them (86%) reported that computers were used in math lessons back when they were in school themselves. The software used was mainly spreadsheets (81%) but also computer algebra (27%) and geometry systems.

When asked for their attitudes and believes about computers the results were as might be expected after this: Almost all of them said that computers should be used and math education and most of them (68%) were sure that computers have a positive influence on the students' motivation.

Given this, the answers to the question „Give examples were computers are useful for mathematics in general or for applications of mathematics" were a bit surprising: 16% gave no answer and 19% mentioned doing fast calculations, but without examples. In the next lesson I took the opportunity to ask the students what calculations they had in mind. The only answer was that "fast computers have the advantage that the addition is done quickly when buying a lot of products in the super market." Well, it is an interesting exercise to figure out if spending a billion dollar at a super market makes up a calculation that takes more than 1 second on a modern computer. But to be serious again, this answer shows that even students who have seen the use of computers in math lessons have no idea why computers boost the influence of mathematics in the tremendous way it does. Another 11% of the students mentioned that computers can be used to calculate complex formula, but again they had no idea about what these formulas could be or where they might occur.

Similarly disappointing were the answers to the question "Did you ever feel that you had a gain by doing math on the computers.": 65% no, 8% yes, during a course on numerical analysis, 8% yes, in the geometry lectures.

Summing up these findings one may wonder if it is inconsistent that so many students had used math software in math lessons and nevertheless had no real idea about why math+computers are such a powerful combination. However, there is an easy explanation: Most of the time computers are used in math lessons, their purpose is to support mathematical activities that were invented before the use of computers. A typical example is given by dynamic geometry systems applied to the problem to find a line tangent to two circles. This is a problem that was historically solved by ruler and compass construction. There are other ways to solve this problem, e.g. calculating. Calculating was made very easy by computers and thus solving the problem by calculation can be considered to be the natural way to use a computer on this problem. However, in math lessons, we hide these calculations deep inside the geometry program and work on the level of synthetic geometry. Much of computer use simply has the aim to illustrate ideas and techniques that were adequate in the pre-computer era. This view is supported by the data from the survey: 85% suggested that computers are useful for mathematics because they allow to visualize graphs or (to a lesser extent) other mathematical objects.

To prevent misunderstanding: I do not say that using dynamic geometry systems or using a computer to visualize graphs is a bad idea. Not at all! We all know how much beautiful mathematics can be done that way and that many students appreciate it. The point is that this is important, but it is not the whole story. If we restrict computer use to that kind of activity, students are actively guided to a problematic view about the relationship between mathematics and computers.

What is needed is a modification of the curriculum that gives students insight into the modern use of computers as specific mathematical tool. Of course, the subject is extremely broad ranging from cryptography to weather forecast. Therefore one should identify some fundamental ideas of computer based mathematics (CBM). They may mediate in some sense between fundamental ideas of mathematics and of computer science. Here is an ad hoc collected list:

- Discretization with linearization: Especially in time based simulations, the continuum of time (and eventually space) is divided in small pieces so that change between them can be viewed to be linear. Governed by this idea are (among others) solution to differential equations and numerical integration.
- Search guided by cost function
- Symbolizing: Problems can often be reduced by casting them into a symbolic form.
- Simulation can compensate for a lack of theory

Given the broad field of applications this is a surprisingly short list. In fact many methods rest on common principles. This is good news for the attempts to bring these ideas to school.

Examples of Tasks that promote the fundamental ideas of CBM

The following examples are selected in way that prototypically shows how the fundamental ideas of CBM mentioned above can be brought to the classroom.

Optimization

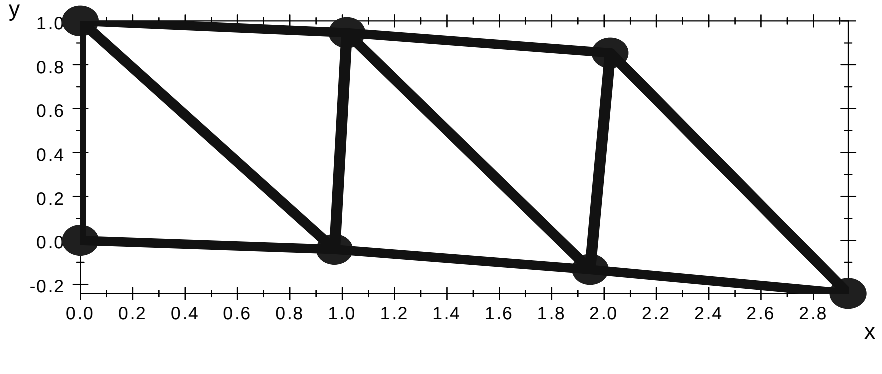
In school mathematics finding extremal values is linked very closely to calculus. I don't want to argue that this link is unimportant, but I think it should be complemented by the fundamental idea of search. For example, given a function in two variables like $f(x,y)=x^2+2x+y^2+5y+xy/2+1$ it is very easy to calculate function values. And then, it is easy to look for values of x and y that yield small function values. Students doing this with f defined in a program or on a calculator so that evaluation is fast for them will almost for sure discover some search strategy how to make the function value smaller, and they will – without doubt – understand that a computer can do this search for them. After half an hour they have an idea about what numerical multi-dimensional minimization is and then they can use it to model situations by defining functions to be minimized. The technological basis can either be a computer algebra system (e.g. using the lbfgs package in the free Maxima system), a programming language with minimization code (e.g. Python with the Scipy library (both free as well)) or a spreadsheet like Excel. Excel contains the solver utility, that allows to minimize the value of calculated cell very easily. One simply has to select the cells that may be changed and the cell to be minimized.

With this technology at hand students can answer questions like this:

What parameter values in a function should be used to fit given data. For example, students can fit a circle $(x-x_0)^2+(y-y_0)^2=r^2$ to a number of points by minimizing the sum of defect squares.

What is the shape of a fast slide between two points (the brachistochrone problem)?

Fig. 1: A crane bended by a heavy load



What configuration will a set of springs take at rest when a given force is applied. A real world application is the deformation of a crane under a load (see Fig. 1).

The brachistochrone example can be solved in a computer algebra system by the following small code. The slide is approximated by a polygon line with 50 points:

```
n:=50                              // Number of points
ya:=0.0                            // Start in point (0,ya)
xe:=3; ye:=-1                      // End in point (xe,ye)
v:= y ->  sqrt(-2*9.81*y)          // Velocity in height y
ttime:=sum( 1/v(0.5*(y[i]+y[i+1]))*
             sqrt((y[i]-y[i+1])^2+(xe/n)^2),  i=0..n-1)
// ttime=total time= Sum over all times in the intervals t= 1/v * s
f:=subs(ttime, y[0]=ya, y[n]=ye) // incorporate end point values
ys:=NEWTON(term,[y[1],…,y[n]],[-1,…,-1])  // find optimum
```

The result plotted as line is shown in Fig. 2.

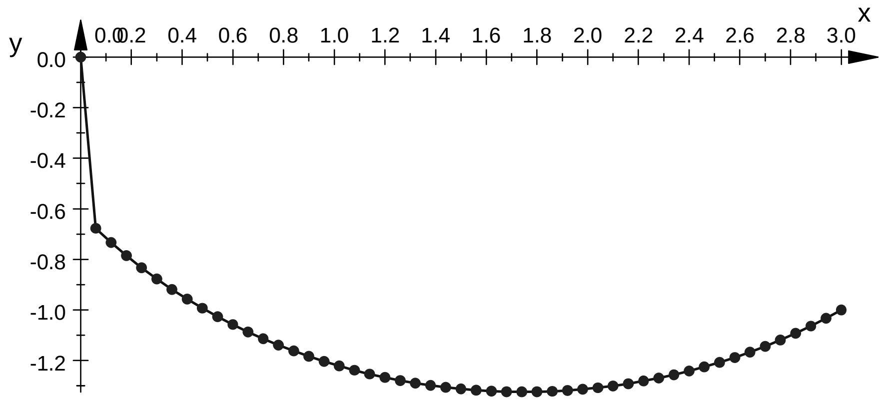


Fig 2: A numerical approximation of a brachistochrone

A slight generalization is constraint optimization. With this technique, only a subspace that is defined by a set of equations is searched for the minimum. A simple example is that of hanging chain.

Again, the method of discretization is important: The chain is approximated by points $(x_i, y_i), i = 0..n$. The quantity to minimize is its energy which is the sum of the energy of its segments and the energy of each segment is proportional to its length (which is in turn proportional to its mass) and its middle height: $E = \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \cdot \frac{y_i + y_{i+1}}{2}$

A constraint results because the chain has a fixed length: $L = \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$.

Either one uses a specializes optimization algorithm for constrained problems or one adds a penalty term to the objective function (with μ a "large" number, say 100) and minimizes the following function:

$$\tilde{f} = \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \cdot \frac{y_i + y_{i+1}}{2} + \mu \cdot \left(L - \sum_{i=0}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \right)^2$$

The result, as shown in Fig.3 clearly shows that a hanging chain is not a parabola.

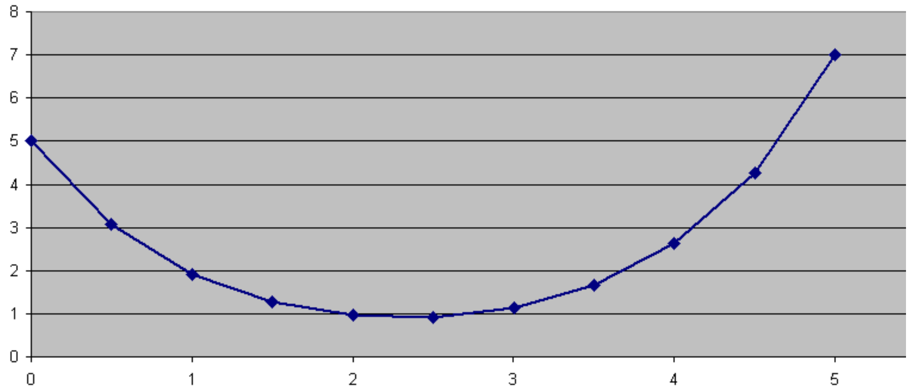


Fig 3: A hanging chain

Summing up, we see that working with optimization algorithms gives students opportunities to work with expressions, functions, equations, to model real world situations, to apply the fundamental ideas of applied mathematics and to see how computers' calculation power can be used to solve nontrivial problems.

Simulating Heat propagation with a spreadsheet

In this section we shift from optimization to differential equations but in a form that takes out most of the analytic obstacles.

A simple experiment shows that the temperature of some hot object decreases and its temperature approaches the temperature of the environment. Data can easily be collected and entered into a spreadsheet. While Temperature is a function of the continuous independent variable time t, measurement introduces a discretization and this can be carried over to the model. Between points in time t_{i+1} and t_i the temperature decreases: $T_{i+1} = T_i - \text{cooling down}$. The cooling down will be large if the time step is large and if the difference to the temperature of the environment T_E is large: $T_{i+1} = T_i - \Delta t \cdot (T_i - T_E) \cdot c$

The parameter is chosen so that the calculated temperature fits the experimental data well.

Now we shift interest from one body to two connected bodies, one hot, the other cool. Similar consideration as above show that now the temperature difference to the neighbor is relevant: $T_{\text{neq}} = T_{old} + (T_{\text{neighbor}} - T_{\text{old}}) \cdot k$. Again, this can be calculated easily in a spreadsheet or any programming language. And it opens the possibility to look at a really nontrivial example: The heat propagation in a rod, e.g. the metal of a pan standing on a hot oven.

Now space is discretized, i.e. the rod is made up of cells that interchange energy with its neighbors:

T1	T2	T3	...	Tn

The change of energy of the i-th segment is: $\Delta E_i = k \cdot (T_{i+1} - T_i) \cdot \Delta t + k \cdot (T_{i-1} - T_i) \cdot \Delta t$.

The development in time is described by a second index j. As above: $T_{i,j+1} = T_{i,j} + \frac{1}{C} \cdot \Delta E_i$. Putting this together and setting $c = \frac{k}{C} \cdot \Delta t$ yields $T_{i,j+1} - T_{i,j} = c \cdot \left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right)$ which is easy to calculate. At time j=0 all initial temperatures T_{i0} must be set and at the left and right border the temperature must be pre-described for all time steps (e.g. a hot and a cold end of the rod). Then all middle values can be calculated uniquely.

This example shows that with rather basic calculations a nontrivial example of a process in space-time can be calculated. This provides students with a prototypical idea about how similar processes can be calculated, an example being weather forecast. It is obvious, that a weather model must include much more details (three-dimensional space, not only temperature is important, not only heat transfer but also convection, etc.) but nevertheless the fundamental ideas of CBM are the same.

Digital Image processing

Digital images are encoded as a matrix of pixels. For gray scale images each pixel is characterized by a single number, its brightness. Thus a gray scale digital image is exactly the same as a matrix in mathematics. One may apply a function to each entry to increase or decrease brightness, contrast or even to turn an image to its negative. This involves only easy algebra. However, usually one has to write programs to do this. To eliminate this difficulty I wrote some Java applets that are accessible on my webpage (http://www.math.uni-frankfurtde/~oldenbur (only in German)) overcome this problem. Students can perform various operations by specifying the mathematics transformation functions. Examples are shown below in Fig. 4 and 5.

Studying these image processing algorithms introduces students to computer based applications of mathematics that brings out all the fundamental ideas described above.

Fig 4: An Applet to transform the brightness of pixels according to a function

Fig 5: Example of a calculated local displacement

Conclusion

The examples presented here can only give a rough idea about what can be done in math education in high school when the computational power of today's computers is utilized as a problem solving tool. Almost all applications can be boiled down to simple forms that illustrate the principles and fundamental ideas of CBM. We believe that this should give students better insight into the role of math and computers in our modern society and we expect that this influence could be demonstrated empirically by observing a change in the students' mathematical belief systems. Although this sketches a plan for future research, it should be pointed out that the topics given here and some others have already been taught successfully in high school.

Literature

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